

NEW EXTENSIONS OF POPOVICIU'S INEQUALITY

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Dedicated to the memory of T. Popoviciu.

ABSTRACT. Popoviciu's inequality is extended to the framework of h -convexity and also to convexity with respect to a pair of quasi-arithmetic means. Several applications are included.

1. INTRODUCTION

Fifty years ago Tiberiu Popoviciu [23] published the following striking characterization of convex functions:

Theorem 1. *A real-valued continuous function f defined on an interval I is convex if and only if it verifies the inequality*

$$(Pop) \quad \frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x + y + z}{3}\right) \geq \frac{2}{3} \left(f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right) \right)$$

whenever $x, y, z \in I$.

He also noticed that inequality (Pop) has higher order analogues for each finite string of points (of length greater than or equal to 3). In [23], only the unweighted case was discussed, but Popoviciu's argument covers the weighted case as well.

Popoviciu's result has received a great deal of attention and many improvements and extensions have been obtained. The interested reader may consult the books of Mitrinović [11], Niculescu and Persson [16] and Pečarić, Proschan and Tong [21], as well as the recent papers by Niculescu and his collaborators [4], [10], [13], [16], [17], [18], [19] and [20].

Two easy extensions of Popoviciu's inequality that escaped unnoticed refer to the case of convex functions with values in a Banach lattice and that of semiconvex functions (i.e., of the functions that become convex after the addition of a suitable smooth function). Using the phenomenon of semiconvexity one can state a Popoviciu type inequality for all functions of class C^2 :

Proposition 1. *Suppose that $f \in C^2([a, b])$ and put*

$$M = \sup \{f''(x) : x \in [a, b]\} \text{ and } m = \inf \{f''(x) : x \in [a, b]\}.$$

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Then

$$\begin{aligned} \frac{M}{36} ((x-y)^2 + (y-z)^2 + (z-x)^2) &\geq \\ \frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) - \frac{2}{3} \left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right) \\ &\geq \frac{m}{36} ((x-y)^2 + (y-z)^2 + (z-x)^2) \end{aligned}$$

for all $x, y, z \in [a, b]$.

Indeed, under the assumptions of Proposition 1, both functions $\frac{M}{2}x^2 - f(x)$ and $f(x) - \frac{m}{2}x^2$ are convex and Theorem 1 applies. The variant of Proposition 1 for strongly convex functions, that is for those functions f such that $f - \frac{C}{2}x^2$ is convex for a suitable $C > 0$) can be deduced in the same manner:

$$\begin{aligned} \frac{f(x) + f(y) + f(z)}{3} + f\left(\frac{x+y+z}{3}\right) - \frac{2}{3} \left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right) \\ \geq \frac{C}{36} ((x-y)^2 + (y-z)^2 + (z-x)^2). \end{aligned}$$

Since $e^x \geq \frac{1}{2}x^2$ for $x \geq 0$, this fact yields the inequality

$$\frac{a+b+c}{3} + \sqrt[3]{abc} - \frac{2}{3} (\sqrt{ab} + \sqrt{bc} + \sqrt{ca}) \geq \frac{1}{36} \left(\log^2 \frac{a}{b} + \log^2 \frac{b}{c} + \log^2 \frac{c}{a} \right),$$

for all $a, b, c \geq 1$.

The aim of the present paper is to discuss Popoviciu's inequality in the context of generalized convexity.

The next section deals with the case of convexity with respect to a pair of means. See Definition 1 below for details. Theorem 2 states the analogue of Popoviciu's inequality in the context of quasi-arithmetic means, and its usefulness is illustrated by the case of the hypergeometric function and the volume function of the unit ball in L^P spaces of dimension n . A counter-example shows that we cannot expect a full extension of Popoviciu's inequality to the case of arbitrary convex functions with respect to a pair of means.

Section 3 deals with the case of h -convex functions in the sense of Varošanec [24]. We end our paper by noticing the availability of Popoviciu's inequality in the general framework of h -Jensen pairs of functions.

2. THE CASE OF CONVEX FUNCTIONS RELATIVE TO A PAIR OF MEANS

Convexity relative to a pair of means was first considered by Aumann [3] in 1933, but its serious investigation started not until the 90s. By a *mean* on an interval I we understand any function $M : I \times I \rightarrow \mathbb{R}$ such that

$$\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}$$

for all $x, y \in I$. The most used class of means is that of quasi-arithmetic means, which are associated to a continuous and strictly monotonic function $\varphi : I \rightarrow \mathbb{R}$ by the formula

$$\mathfrak{M}_\varphi(x, y) = \varphi^{-1} \left(\frac{\varphi(x) + \varphi(y)}{2} \right), \text{ for } x, y \in I.$$

A particular case is that of power means of order $p \in \mathbb{R}$,

$$M_p(x, y) = \begin{cases} \min \{x, y\} & \text{if } p = -\infty \\ \left(\frac{x^p + y^p}{2} \right)^{1/p} & \text{if } p \neq 0 \\ \sqrt{xy} & \text{if } p = 0 \\ \max \{x, y\} & \text{if } p = \infty, \end{cases}$$

which corresponds to the function $\varphi(x) = x^p$, if $p \in \mathbb{R} \setminus \{0\}$ and $\varphi(x) = \log x$, if $p = 0$. Notice that

$$\begin{aligned} M_{-1} &= H \text{ (the harmonic mean)} \\ M_0 &= G \text{ (the geometric mean)} \\ M_1 &= A \text{ (the arithmetic mean).} \end{aligned}$$

Remarkably, the quasi-arithmetic means \mathfrak{M}_φ admit natural extensions to the case of an arbitrary finite family of points x_1, \dots, x_n endowed with weights $\lambda_1, \dots, \lambda_n$ of total mass 1,

$$\mathfrak{M}_\varphi(x_1, \dots, x_n; \lambda_1, \dots, \lambda_n) = \varphi^{-1} \left(\sum_{k=1}^n \lambda_k \varphi(x_k) \right).$$

In order to simplify the notation, we put $\mathfrak{M}_\varphi(x_1, \dots, x_n; 1/n, \dots, 1/n) = \mathfrak{M}_\varphi(x_1, \dots, x_n)$.

Definition 1. *Given a pair of intervals I and J endowed respectively with the means M and N , a function $f : I \rightarrow J$ is called (M, N) -convex if it is continuous and*

$$((M, N)) \quad f(M(x, y)) \leq N(f(x), f(y)) \quad \text{for all } x, y \in I.$$

The analogue of Jensen's inequality works in the case of $(\mathfrak{M}_\varphi, \mathfrak{M}_\psi)$ -convex functions, so that for such functions we have

$$f(\mathfrak{M}_\varphi(x_1, \dots, x_n; \lambda_1, \dots, \lambda_n)) \leq \mathfrak{M}_\psi(f(x_1), \dots, f(x_n); \lambda_1, \dots, \lambda_n)$$

for all $x_1, \dots, x_n \in I$ and $\lambda_1, \dots, \lambda_n \in [0, 1]$ with $\sum \lambda_k = 1$.

Clearly, the usual convex functions represent the case of (A, A) -convex functions, while the log-convex functions are the same with (A, G) -convex functions.

The importance and significance of other classes of generalized convex functions such as of (G, A) -convex functions, (G, G) -convex functions, (H, A) -convex functions etc. is discussed in the book [15] and the paper of Anderson, M.K. Vamanamurthy, M. Vuorinen [2].

Not all important means are quasi-arithmetic. Two examples are the *logarithmic mean*,

$$L(a, b) = \begin{cases} \frac{a-b}{\ln a - \ln b} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}$$

and the *identric mean*,

$$I(a, b) = \begin{cases} \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \\ a & \text{if } a = b. \end{cases}$$

The theory of $(\mathfrak{M}_\varphi, \mathfrak{M}_\psi)$ -convex functions can be deduced from the theory of usual convex functions.

Lemma 1. (J. Aczel [1]). *Let φ and ψ be two strictly monotonic functions defined respectively on the intervals I and J , and let $f : I \rightarrow J$ be an arbitrary function.*

If ψ is strictly increasing, then f is $(\mathfrak{M}_\varphi, \mathfrak{M}_\psi)$ -convex/concave if and only if $\psi \circ f \circ \varphi^{-1}$ is convex/concave on $\varphi(I)$ in the usual sense.

If ψ is strictly decreasing, then f is $(\mathfrak{M}_\varphi, \mathfrak{M}_\psi)$ -convex/concave if and only if $\psi \circ f \circ \varphi^{-1}$ is concave/convex on $\varphi(I)$ in the usual sense.

Theorem 1 yields the following extension of Popoviciu's inequality:

Theorem 2. *Suppose that $f : I \rightarrow J$ is an $(\mathfrak{M}_\varphi, \mathfrak{M}_\psi)$ -convex function. If ψ is strictly increasing, then*

$$\begin{aligned} \mathfrak{M}_\psi(\mathfrak{M}_\psi(f(x), f(y), f(z)), f(\mathfrak{M}_\varphi(x, y, z))) \\ \geq \mathfrak{M}_\psi(f(\mathfrak{M}_\varphi(x, y)), f(\mathfrak{M}_\varphi(y, z)), f(\mathfrak{M}_\varphi(z, x))) \end{aligned}$$

for all $x, y, z \in I$.

The inequality works in the reverse sense if the function ψ is strictly decreasing.

Proof. By Lemma 1 the function $\psi \circ f \circ \varphi^{-1}$ is convex on the interval $\varphi(I)$ so that one can apply Popoviciu's inequality to it relative to the points $a = \varphi(x)$, $b = \varphi(y)$ and $c = \varphi(z)$. Then

$$\begin{aligned} & \frac{(\psi \circ f \circ \varphi^{-1})(\varphi(x)) + (\psi \circ f \circ \varphi^{-1})(\varphi(y)) + (\psi \circ f \circ \varphi^{-1})(\varphi(z))}{3} \\ & \quad + (\psi \circ f \circ \varphi^{-1})\left(\frac{\varphi(x) + \varphi(y) + \varphi(z)}{3}\right) \\ & \geq \frac{2}{3} \left((\psi \circ f \circ \varphi^{-1})\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + (\psi \circ f \circ \varphi^{-1})\left(\frac{\varphi(y) + \varphi(z)}{2}\right) \right. \\ & \quad \left. + (\psi \circ f \circ \varphi^{-1})\left(\frac{\varphi(z) + \varphi(x)}{2}\right) \right), \end{aligned}$$

that is,

$$\begin{aligned} & \frac{1}{2} \left(\frac{\psi(f(x)) + \psi(f(y)) + \psi(f(z))}{3} + \psi(f(\mathfrak{M}_\varphi(x, y, z))) \right) \\ & \geq \frac{\psi(f(\mathfrak{M}_\varphi(x, y))) + \psi(f(\mathfrak{M}_\varphi(y, z))) + \psi(f(\mathfrak{M}_\varphi(z, x)))}{3}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\psi(f(x)) + \psi(f(y)) + \psi(f(z))}{3} &= \psi \left(\psi^{-1} \left(\frac{\psi(f(x)) + \psi(f(y)) + \psi(f(z))}{3} \right) \right) \\ &= \psi(\mathfrak{M}_\psi(f(x), f(y), f(z))), \end{aligned}$$

and the proof ends by applying ψ^{-1} to both sides. \square

This result can be extended to the case of an arbitrary finite family of points and weighted quasi-arithmetic means, but the details are tedious and will be omitted.

Example 1. *The Gaussian hypergeometric function (of parameters $a, b, c > 0$) is defined via the formula*

$$F(x) = {}_2F_1(x; a, b, c) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} x^n \quad \text{for } |x| < 1,$$

where $(a, n) = a(a+1) \cdots (a+n-1)$ if $n \geq 1$ and $(a, 0) = 1$. Anderson, Vamanamurthy and Vuorinen [2] proved that if $a + b \geq c > 2ab$ and $c \geq a + b - 1/2$, then the function $1/F(x)$ is concave on $(0, 1)$. This implies

$$F\left(\frac{x+y}{2}\right) \leq \frac{1}{\frac{1}{2}\left(\frac{1}{F(x)} + \frac{1}{F(y)}\right)} \quad \text{for all } x, y \in (0, 1),$$

whence it follows that the hypergeometric function is (A, H) -convex. Taking into account that the harmonic mean is a quasi-arithmetic mean corresponding to the strictly decreasing function $\frac{1}{x}$, we infer that F verifies the following analogue of Popoviciu's inequality:

$$\frac{1}{\frac{1}{2}\left(\frac{1}{3}\left(\frac{1}{F(x)} + \frac{1}{F(y)} + \frac{1}{F(z)}\right) + \frac{1}{F\left(\frac{x+y+z}{3}\right)}\right)} \leq \frac{1}{\frac{1}{3}\left(\frac{1}{F\left(\frac{x+y}{2}\right)} + \frac{1}{F\left(\frac{y+z}{2}\right)} + \frac{1}{F\left(\frac{z+x}{2}\right)}\right)},$$

equivalently,

$$\begin{aligned} \frac{1}{2}\left(\frac{1}{3}\left(\frac{1}{F(x)} + \frac{1}{F(y)} + \frac{1}{F(z)}\right) + \frac{1}{F\left(\frac{x+y+z}{3}\right)}\right) \\ \geq \frac{1}{3}\left(\frac{1}{F\left(\frac{x+y}{2}\right)} + \frac{1}{F\left(\frac{y+z}{2}\right)} + \frac{1}{F\left(\frac{z+x}{2}\right)}\right). \end{aligned}$$

Example 2. D. Borwein, J. Borwein, G. Fee and R. Girgensohn [5] proved that the volume $V_n(p)$ of the convex body $\mathcal{E} = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$ is an (H, G) -concave function on $[1, \infty)$. More precisely, given $\alpha > 1$, the function

$$V_\alpha(p) = 2^\alpha \frac{\Gamma^\alpha(1 + 1/p)}{\Gamma(1 + \alpha/p)}$$

verifies the inequality

$$V_\alpha^{1-\lambda}(p)V_\alpha^\lambda(q) \leq V_\alpha\left(\frac{1}{\frac{1-\lambda}{p} + \frac{\lambda}{q}}\right)$$

for all $p, q > 0$ and $\lambda \in [0, 1]$. In this case, Popoviciu's inequality becomes

$$\begin{aligned} \sqrt[3]{V_\alpha(p)V_\alpha(q)V_\alpha(r)} \cdot V_\alpha\left(\frac{1}{\frac{1}{3}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)}\right) \\ \geq \sqrt[3]{V_\alpha\left(\frac{1}{\frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)}\right) \cdot V_\alpha\left(\frac{1}{\frac{1}{2}\left(\frac{1}{q} + \frac{1}{r}\right)}\right) \cdot V_\alpha\left(\frac{1}{\frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right)}\right)}. \end{aligned}$$

A natural question is whether Popoviciu's inequality works for an arbitrary (M, N) -convex function.

We shall see that the answer is negative. Indeed, the log-convex functions are also (A, L) -convex, because they verify the inequalities

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \exp\left(\frac{1}{b-a} \int_a^b \log f(x) dx\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq L(f(a), f(b)) \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

See [14].

The logarithmic mean was extended to the case of an arbitrary finite family of points by Neuman in his paper [12]. An argument that Neuman's extension is the "right" one can be found in [14]. For triplets, the logarithmic mean is given by the formula

$$L(a, b, c) = \frac{2a}{\log \frac{a}{b} \log \frac{a}{c}} + \frac{2b}{\log \frac{b}{a} \log \frac{b}{c}} + \frac{2c}{\log \frac{c}{a} \log \frac{c}{b}}.$$

The analogue of Popoviciu's inequality in the case of (A, L) -convex functions should be

$$\begin{aligned} \frac{L(f(x), f(y), f(z)) - f\left(\frac{x+y+z}{3}\right)}{\log L(f(x), f(y), f(z)) - \log f\left(\frac{x+y+z}{3}\right)} \\ \geq L\left(f\left(\frac{x+y}{2}\right), f\left(\frac{y+z}{2}\right), f\left(\frac{z+x}{2}\right)\right), \end{aligned}$$

for all x, y, z belonging to the domain of f . However this does not work even in the case of the Gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0,$$

that is known to be log-convex (see [15], Theorem 2.2.1, pp. 68-69). The Gamma function has a minimum at 1.461632..., so we will search around this point.

Put

$$\begin{aligned} E(x; y; z) &= \frac{L(\Gamma(x), \Gamma(y), \Gamma(z)) - \Gamma\left(\frac{x+y+z}{3}\right)}{\log L(\Gamma(x), \Gamma(y), \Gamma(z)) - \log \Gamma\left(\frac{x+y+z}{3}\right)} \\ &\quad - L\left(\Gamma\left(\frac{x+y}{2}\right), \Gamma\left(\frac{y+z}{2}\right), \Gamma\left(\frac{z+x}{2}\right)\right) \end{aligned}$$

for $x, y, z > 0$. A simple computation shows that

$$E(1.40; 1.46; 1.47) = 65.92090117 - 108.64 < 0$$

while

$$E(0.30; 0.34; 0.35) = 2.711369453 - 2.709270 > 0.$$

Therefore Popoviciu's inequality does not always work for (M, N) -convex functions.

3. THE CASE OF h -CONVEX FUNCTIONS

In 2007, Varošaneć [24] introduced a class of generalized convex functions that brings together several important classes of functions.

In order to enter into the details we have to fix a function $h : (0, 1) \rightarrow (0, \infty)$ such that

$$(3.1) \quad h(1 - \lambda) + h(\lambda) \geq 1 \text{ for all } \lambda \in (0, 1).$$

As above, I will denote an interval.

Definition 2. A function $f : I \rightarrow \mathbb{R}$ is called h -convex if

$$f((1 - \lambda)x + \lambda y) \leq h(1 - \lambda)f(x) + h(\lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in (0, 1)$.

The role of the condition (3.1) is to assure that the function identically 1 is h -convex.

The usual convex functions represent the particular case of Definition 2, where h is the identity function.

The h -convex functions corresponding to the case $h(\lambda) = \lambda^s$ (for a suitable $s \in (0, 1]$) are the s -convex functions in the sense of Breckner [6]. Their systematic study can be found in the papers of Hudzik and Maligranda [9] and Pinheiro [22].

An example of an s -convex function (for $0 < s < 1$) is given by the formula

$$f(t) = \begin{cases} a & \text{if } t = 0 \\ bt^s + c & \text{if } t > 0 \end{cases}$$

where $b \geq 0$ and $0 \leq c \leq a$. In particular, the function t^s is s -convex on $[0, \infty)$ if $0 < s < 1$.

The nonnegative h -convex functions corresponding to the case $h(\lambda) = \frac{1}{\lambda}$ are the convex functions in the sense of Godunova-Levin [8]. They verify the inequality

$$f((1 - \lambda)x + \lambda y) \leq \frac{f(x)}{1 - \lambda} + \frac{f(y)}{\lambda}$$

for all $x, y \in I$ and $\lambda \in (0, 1)$. Every nonnegative monotonic function (as well as every nonnegative convex function) is convex in the sense of Godunova-Levin.

The h -convex functions corresponding to the case $h(\lambda) \equiv 1$ are the P -convex functions in the sense of Dragomir, Pečarić and Persson [7]. They verify inequalities of the form

$$f((1 - \lambda)x + \lambda y) \leq f(x) + f(y)$$

for all $x, y \in I$ and $\lambda \in (0, 1)$.

One can state the following analogue of Popoviciu's inequality in the case of h -convex functions.

Theorem 3. If h is concave, then every nonnegative h -convex function $f : I \rightarrow \mathbb{R}$ verifies the inequality

$$\begin{aligned} (hPop) \quad \max\{h(1/2), 2h(1/4)\} (f(x) + f(y) + f(z)) + 2h(3/4)f\left(\frac{x+y+z}{3}\right) \\ \geq f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \end{aligned}$$

for all $x, y, z \in I$.

Proof. Without loss of generality we may assume that $x \leq y \leq z$. If $y \leq (x + y + z)/3$, then

$$(x + y + z)/3 \leq (x + z)/2 \leq z \quad \text{and} \quad (x + y + z)/3 \leq (y + z)/2 \leq z,$$

which yields two numbers $s, t \in [0, 1]$ such that

$$\begin{aligned} \frac{x + z}{2} &= s \cdot \frac{x + y + z}{3} + (1 - s) \cdot z \\ \frac{y + z}{2} &= t \cdot \frac{x + y + z}{3} + (1 - t) \cdot z. \end{aligned}$$

Summing up, we get $(x + y - 2z)(s + t - 3/2) = 0$. If $x + y - 2z = 0$, then necessarily $x = y = z$, and the inequality (*hPop*) is clear. If $s + t = 3/2$, by summing up the following three inequalities

$$\begin{aligned} f\left(\frac{x + z}{2}\right) &\leq h(s) \cdot f\left(\frac{x + y + z}{3}\right) + h(1 - s) \cdot f(z) \\ f\left(\frac{y + z}{2}\right) &\leq h(t) \cdot f\left(\frac{x + y + z}{3}\right) + h(1 - t) \cdot f(z) \\ f\left(\frac{x + y}{2}\right) &\leq h(1/2) \cdot f(x) + h(1/2) \cdot f(y). \end{aligned}$$

we get

$$\begin{aligned} &f\left(\frac{x + y}{2}\right) + f\left(\frac{y + z}{2}\right) + f\left(\frac{z + x}{2}\right) \\ &\leq (h(s) + h(t)) \cdot f\left(\frac{x + y + z}{3}\right) \\ &\quad + h(1/2) \cdot f(x) + h(1/2) \cdot f(y) + (h(1 - s) + h(1 - t)) \cdot f(z) \\ &\leq h(1/2) \cdot f(x) + h(1/2) \cdot f(y) + 2h(1/4) \cdot f(z) + 2h(3/4) f\left(\frac{x + y + z}{3}\right) \\ &\leq \max\{h(1/2), 2h(1/4)\} (f(x) + f(y) + f(z)) + 2h(3/4) f\left(\frac{x + y + z}{3}\right), \end{aligned}$$

and the inequality (*hPop*) is also clear.

The case where $(x + y + z)/3 < y$ can be treated in a similar way. \square

As an application of Theorem 3 let us consider the case of the function $t^{1/2}$ (which is s -convex for $s = 1/2$). Then $h(t) = t^{1/2}$, $\max\{h(1/2), 2h(1/4)\} = 1$ and $2h(3/4) = \sqrt{3}$, which yields

$$\begin{aligned} x^{1/2} + y^{1/2} + z^{1/2} + \sqrt{3} \left(\frac{x + y + z}{3}\right)^{1/2} \\ \geq \left(\frac{x + y}{2}\right)^{1/2} + \left(\frac{y + z}{2}\right)^{1/2} + \left(\frac{z + x}{2}\right)^{1/2} \end{aligned}$$

for all $x, y, z \geq 0$.

We end our paper with another Popoviciu type inequality for h -convex functions.

The basic ingredient is the Jensen-type inequality for the h -convex functions,

$$f\left(\frac{x_1 + \dots + x_n}{n}\right) \leq h\left(\frac{1}{n}\right) (f(x_1) + \dots + f(x_n))$$

valid for arbitrary finite strings of points x_1, \dots, x_n under the additional hypothesis that h is supermultiplicative in the sense that $h(xy) \geq h(x)h(y)$ for all x, y . See [24], Theorem 19. When f is h -concave and h is submultiplicative, the Jensen inequality takes the form

$$f\left(\frac{x_1 + \dots + x_n}{n}\right) \geq h\left(\frac{1}{n}\right) (f(x_1) + \dots + f(x_n)).$$

Theorem 4. *i) If h is supermultiplicative, with $h(1/3) < 1$, and $f : I \rightarrow \mathbb{R}$ is an h -convex function, then*

$$\begin{aligned} f(x) + f(y) + f(z) - f\left(\frac{x+y+z}{3}\right) \\ \geq \frac{1-h(1/3)}{2h(1/2)} \left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right) \end{aligned}$$

for all $x, y, z \in I$.

ii) If h is submultiplicative, with $h(1/3) > 1$, and $f : I \rightarrow \mathbb{R}$ is an h -concave function, then

$$\begin{aligned} f\left(\frac{x+y+z}{3}\right) - (f(x) + f(y) + f(z)) \\ \geq \frac{h(1/3)-1}{2h(1/2)} \left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right). \end{aligned}$$

Proof. *i)* In this case,

$$\begin{aligned} f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) &\leq 2h(1/2) (f(x) + f(y) + f(z)) \\ &= \frac{2h(1/2)}{1-h(1/3)} (f(x) + f(y) + f(z)) - \frac{2h(1/2)}{1-h(1/3)} h(1/3) (f(x) + f(y) + f(z)) \\ &\leq \frac{2h(1/2)}{1-h(1/3)} (f(x) + f(y) + f(z)) - \frac{2h(1/2)}{1-h(1/3)} f\left(\frac{x+y+z}{3}\right) \\ &= \frac{2h(1/2)}{1-h(1/3)} \left(f(x) + f(y) + f(z) - f\left(\frac{x+y+z}{3}\right) \right). \end{aligned}$$

ii) Similarly,

$$\begin{aligned} f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) &\geq 2h(1/2) (f(x) + f(y) + f(z)) \\ &= \frac{2h(1/2)h(1/3)}{h(1/3)-1} (f(x) + f(y) + f(z)) - \frac{2h(1/2)}{h(1/3)-1} (f(x) + f(y) + f(z)) \\ &\geq \frac{2h(1/2)}{h(1/3)-1} \left(f\left(\frac{x+y+z}{3}\right) - (f(x) + f(y) + f(z)) \right). \end{aligned}$$

□

As an application of Theorem 3 let us consider the case of the function $t^{1/2}$ (which is s -convex for $s = 1/2$). Then $h(t) = t^{1/2}$ and $h(1/3) = (1/3)^{1/2} = 0.577... < 1$.

Therefore

$$\begin{aligned} x^{1/2} + y^{1/2} + z^{1/2} - \left(\frac{x+y+z}{3} \right)^{1/2} \\ \geq \frac{1 - (1/3)^{1/2}}{2(1/2)^{1/2}} \left[\left(\frac{x+y}{2} \right)^{1/2} + \left(\frac{y+z}{2} \right)^{1/2} + \left(\frac{z+x}{2} \right)^{1/2} \right] \end{aligned}$$

for all $x, y, z \geq 0$.

Last but not least it is worth noticing that Popoviciu's inequality still works in the more general context of h -Jensen pairs (f, g) . These pairs are aimed to satisfy inequalities of the form

$$f((1-\lambda)x + \lambda y) \leq h(1-\lambda)g(x) + h(\lambda)g(y),$$

for all $x, y \in I$ and $\lambda \in (0, 1)$; here I is a common domain of f and g . An inspection of the argument of Theorem 3 easily yields the following result.

Theorem 5. *Let h be concave and (f, g) be an h -Jensen pair of positive functions $f, g: I \rightarrow \mathbb{R}$. Then a Popoviciu type inequality holds:*

$$\begin{aligned} \max \{h(1/2), 2h(1/4)\} (g(x) + g(y) + g(z)) + 2h(3/4)g\left(\frac{x+y+z}{3}\right) \\ \geq f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right). \end{aligned}$$

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